Clustering Partial Lexicographic Preference Trees

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Abstract

In this work, we consider the problem of clustering *partial lexicographic preference trees* (PLP-trees), intuitive and often compact representations of user preferences over multi-valued attributes. Due to the preordering nature of PLP-trees, we define a variant of Kendall's τ distance metric to be used to compute distances between PLP-trees for clustering. To this end, extending the previous work by Li and Kazimipour (Li and Kazimipour 2018), we propose a polynomial time algorithm *PlpDis* to compute such distances, and present empirical results comparing it against the brute-force baseline. Based on *PlpDis*, we use various distance-based clustering methods to cluster PLP-trees learned from a car evaluation dataset. Our experiments show that hierarchical agglomerative nesting (AGNES) is the best choice for clustering PLP-trees, and that the *single-linkage* variant of AGNES is the best fit for clustering large numbers of trees.

Introduction

Understanding the decision patterns of users and modeling their preferences is an important problem in artificial intelligence and has received a lot of attention in recent years. A myriad of models for the task of preference representation have been proposed, at the frontier of which are partial lexicographic preference trees (PLP-trees, for short) (Liu and Truszczynski 2015), and conditional preference networks (CP-nets) (Boutilier et al. 2004). PLP-trees are particularly interesting, for they can succinctly encode a total preorder (i.e., a binary relation that is total, reflective, and transitive) of exponentially many alternatives into the structure of an often compact tree. CP-nets also provide an intuitive encoding with the *ceteris paribus* semantics, but they generally are computationally hard to learn and reason about. For this reason we focus on PLP-trees, which show great potential as effective and explainable models.

PLP-trees are useful for modeling individual preferences. Meanwhile, being able to cluster and reason about collections of preferences is critical to various research areas, such as recommender systems, marketing, and human-centric machine learning. Clustering a set of entities typically relies on computing distances between them. Examples of distance-based metrics are Euclidean distance, Spearman's ρ, and Kendall's τ. Because PLP-trees induce *total preorders* over alternatives, we focus on Kendall's τ.

Clearly, one may use the straightforward brute-force algorithm to compute τ; however, due to the exponentiality of the space of alternatives, this direct method practically is infeasible. Recently, Li and Kazimipour (Li and Kazimipour 2018) proposed a computationally efficient algorithm, called *LpDis*, that utilizes the tree structures to compute τ in time polynomial in the size of the two given trees.

However, their results are limited to *complete* models, trees that require *every* attribute to be present in every branch. On the contrary, PLP-trees allow for missing attributes, thus encoding total preorders, as opposed to total orders, and allow for more concise preference representations. To this end, we study the clustering problem of PLP-trees. At the core of this problem, we investigate the problem of computing Kendall's τ distance and propose a new polynomial time algorithm *PlpDis*, an extension to the *LpDis* algorithm. In the remainder of this paper, we provide necessary preliminaries, define a variant of Kendall's τ for PLP-trees and describe our *PlpDis* algorithm that accommodates partial trees, present results on PLP-tree clustering with *PlpDis* as the distance measure, and point out future research directions.

Partial Lexicographic Preference Trees

Let $\mathcal{V} = \{X_1, ..., X_n\}$ be a set of n categorical *attributes* with each $X_i \in \mathcal{V}$ having a finite domain of values $D_i = \{x_1^i, \ldots, x_{m_i}^i\}$. The *combinatorial domain* $CD(\mathcal{V})$ over $\mathcal{V}$ is the Cartesian product $D_1 \times \ldots \times D_n$. We call elements of $CD(\mathcal{V})$ *alternatives*. A *PLP-tree* over $\mathcal{V}$ is an ordered labeled tree where every non-leaf node 1) is labeled with an attribute X_i from $\mathcal{V}$, 2) has a local preference $>_i$ (a total order over D_i), and 3) has $|D_i|$ outgoing edges ordered from left to right according to $>_i$. Additionally, each attribute from $\mathcal{V}$ appears *at most once* in any branch of the tree. We denote a leaf node with $\square$, indicating a bucket of alternatives.

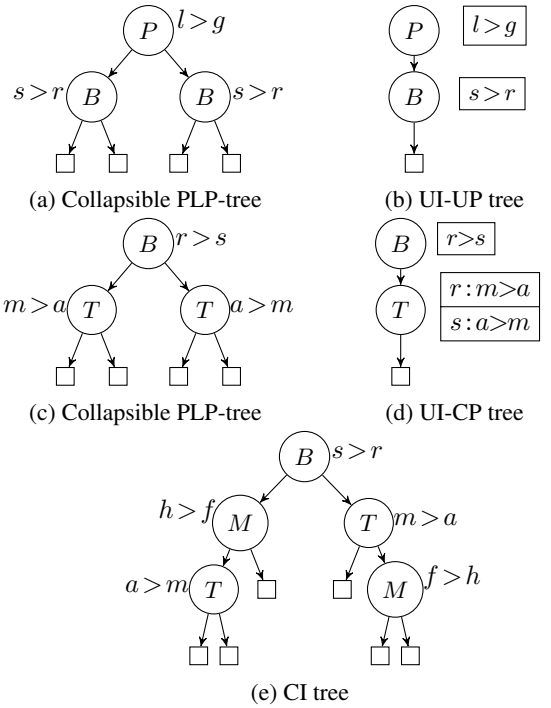
We compute the distance between PLP-trees by considering the *disagreements* between the orders they represent. Let T_1 and T_2 be two PLP-trees (equivalently, two total preorders), and α and β two distinct alternatives from $CD(\mathcal{V})$. The ordering of α and β on T_1 and T_2 either strictly agree, strictly disagree, or partially disagree. Strict agreement oc-

curs when in both T_1 and T_2 either $\alpha \succ \beta$, $\beta \succ \alpha$, or $\alpha \approx \beta$, where $\succ$ means strict preference and $\approx$ means equivalence. Strict disagreement occurs when in T_1 we have $\alpha \succ \beta$ and in T_2 $\beta \succ \alpha$, or vice versa. Partial disagreement occurs when in T_1 $\alpha \approx \beta$ and in T_2 $\alpha \not\approx \beta$ ($\alpha \succ \beta$ or $\beta \succ \alpha$), or vice versa. Examples and detailed discussion on the semantics, classes, and notations of PLP-trees follow in the next three sections.

An Example of PLP-trees

Let us consider a domain of cars specified using four binary attributes: BodyType (B): sedan (s) and sport (r), Make (M): Honda (h) and Ford (f), Price (P): low (l) and high (g), and Transmission (T): automatic (a) and manual (m).

The PLP-tree in Figure 1e bears the following user preferences. The most important attribute for the user is BodyType, for which she prefers sedan to sport. Among sedan cars, the most important attribute is Make and the user prefers Honda over Ford. Similarly, among sport cars, the most important is Transmission and manual is better than automatic for the user. Moreover, among the Honda sedans, the user's most important attribute is Transmission, on which automatic is better than manual. On the other hand, among the automatic sport cars, her most important attribute to consider is Make and she likes Ford more than Honda.



(a) Collapsible PLP-tree (b) UI-UP tree

(c) Collapsible PLP-tree (d) UI-CP tree

(e) CI tree

Figure 1: PLP-trees over the car domain

Semantics of PLP-trees

We see that non-leaf nodes in a PLP-tree are to partition the space of alternatives into groups of them. Non-leaf nodes further down in the PLP-tree refine this partitioning. In other words, given an alternative and a PLP-tree, one may traverse the tree from the root down to some leaf: when at a node, descend to the left child if the alternative has the preferred value on that node, or to the right child, otherwise. A leaf node, therefore, represents a group of alternatives descending to it.

Indexing the leaves from the left most one with 1, 2, ..., we now can define the preference relation induced by a PLP-tree. Let α and β be two alternatives, and $l_T(\alpha)$ and $l_T(\beta)$ their leaf indices in PLP-tree T. We define that α is at least as good as β (denoted $\alpha \succeq_T \beta$) if $l_T(\alpha) \leq l_T(\beta)$, that α is strictly better than β (denoted $\alpha \succ_T \beta$) if $l_T(\alpha) < l_T(\beta)$, and that α is equivalent with β (denoted $\alpha \approx_T \beta$) if $l_T(\alpha) = l_T(\beta)$. Clearly, relation $\succeq_T$ is a total preorder that is total, reflective and transitive.

Classifications of PLP-trees

Exploring the tree structures and the local preferences on nodes, we introduce different classes of PLP-trees. (Our classification of PLP-trees is a slight variant of the proposed by Liu and Truszczynski (Liu and Truszczynski 2015).) When the tree structure is complete, i.e., every level is full, and the nodes per the same level are all labeled by the same attribute, we collapse the tree so that the tree is simplified to a linear path, dramatically reducing the size of the tree. The collapsed tree is called an *unconditional importance* tree, or UI tree. During this collapsing, one question is about what to do with the local preferences. If the preferences on the attributes per the same level are unanimous, the same preference rule then is used to label the attribute in the UI tree. (The resulting tree is called a UI and *unconditional preference* tree, or UI-UP tree, for short.) If, however, the preferences per the same level are different, then a conditional preference table (CPT) is created and used as the label. This resulting tree is called a UI and *conditional preference* tree, or UI-CP tree. The CPT consists of preference rules of form $u : a > b$ that expresses that, conditioned on u, the evaluations of parent attributes, the user prefers a to b on the labeling attribute. Examples of a UI-UP and UI-CP tree and their full versions over the car domain are shown in Figure 1b and Figure 1d, respectively. Other PLP-trees that are not UI trees are called *conditional importance* trees, or CI trees. Notably, given a PLP-tree T of any class and two alternatives α and β, deciding if $\alpha \succ_T \beta$ or $\alpha \approx_T \beta$ can be done in time linear in the size of T.

More Notations in PLP-trees

Let T be a PLP-tree of any class in $\{UI-UP, UI-CP, CI\}$, and n a node in T. We denote by A node n's *ancestor* attributes to be the collection of attributes labeling the ancestors of n, by B node n's *branching* attributes to be the set of attributes labeling those ancestors with multiple children nodes, by P node n's *parent* attributes to be the set of attributes forming the conditions in the CPT at n.

For instance, in Figure 1d, for the node labeled by attribute T, we have $A = \{B\}$, $B = \emptyset$, and $P = \{B\}$. In Figure 1e, for the node labeled by attribute M and preference rule $f > h$, we have $A = \{B, T\}$, $B = \{B, T\}$, and $P = \emptyset$.

We say two nodes from two PLP-trees are *consistent* if they assign the same value to all common branching ancestor attributes. Let $b[B]$ and $b'[B']$ denote the value assignments, b and b', to the branching ancestor attribute sets B and B' for two nodes n and n', respectively. Formally, n and n' are consistent if $b[B \cap B'] = b'[B \cap B']$. Clearly n and n' are consistent if $B \cap B' = \emptyset$.

Distance Between PLP-trees

We see that the distance between PLP-trees involves partial disagreements that are not accounted for in the regular definition of Kendall's τ. Thus, we first define a variant Kendall's τ, called *partial* Kendall's τ, denoted as τ'. Let SD_{T_1,T_2}, and PD_{T_1,T_2} be the set of pairs of alternatives on which T_1 and T_2 strictly disagree and partially disagree, respectively. The metric τ' is then the weighted sum:

$$\tau'(T_1, T_2) = c_1 * |SD_{T_1,T_2}| + c_2 * |PD_{T_1,T_2}|, \quad (1)$$

where c_1 and c_2 are two constant coefficients that may be adjusted based on which of the disagreement type is favored over the other.

Clearly, computing $\tau'(T_1, T_2)$ boils down to counting the pairs in SD_{T_1,T_2} and PD_{T_1,T_2}. We can compute the number of strict disagreements between two PLP-trees, $|SD_{T_1,T_2}|$, using the ***LpDis*** algorithm (Li and Kazimipour 2018). To compute $|SD_{T_1,T_2}|$, for every non-leaf node n in T_1 we traverse every non-leaf node n' in T_2 and accumulate $|SD_{n,n'}|$. Equivalently, we compute $\sum_{n \in T_1, n' \in T_2} |SD_{n,n'}|$. Due to space constraint, we refer the reader to the ***LpDis*** paper for details about the traversal and the equations used to compute $|SD_{n,n'}|$.

We now present the equation for computing $|PD_{T_1,T_2}|$, the novel contribution in our ***PlpDis*** algorithm. To compute $|PD_{T_1,T_2}|$, we extend the ***LpDis*** traversal down to the leaves in both trees and compute the number of alternative pairs decided both at a leaf node in one tree and at a non-leaf node in the other, which we denote by $|P_n \cap P_{n'}|$. Thus, we have $|PD_{T_1,T_2}| = \sum_{n \in T_1, n' \in T_2} |P_n \cap P_{n'}|$, where exactly one of n and n' is a leaf node. To be *decided* at a node means that the preference relation over a pair of alternatives is encoded by that node of the tree. Since leaf nodes always encode the relation $\approx$ and non-leaf nodes encode the relation $\succ$, according to the local preference $>$, computing $|P_n \cap P_{n'}|$ gives the number of partial disagreements. For all distinct pairs of n and n' from both trees (w.l.o.g., n is a leaf), if the attribute labeling n', $V_{n'}$, is not an ancestor of n and the branching attributes in both trees are *consistent*, we have:

$$|P_n \cap P_{n'}| = \binom{|D_{V_{n'}}|}{2} \times \prod_{X \in \breve{A} \setminus \breve{B}} |D_X| \times \prod_{Y \in \mathcal{V} \setminus (\breve{A} \cup \{V_{n'}\})} |D_Y|^2 \quad (2)$$

The first term computes the number of distinct pairs of values in the domain of $V_{n'}$ and the other two terms adjust for possible values of ancestor and descendant (including missing) attributes, respectively.

Clearly, our algorithm ***PlpDis*** to compute $\tau'(T_1, T_2)$ runs in time polynomial in the size of T_1 and T_2. This follows from the fact that in the worst case we compare every pair of nodes between the two trees.

Clustering PLP-trees

With ***PlpDis*** defined, we may now reason about collections of PLP-trees through clustering. Because ***PlpDis*** does not satisfy the triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$, we focus on clustering algorithms that utilize distances between pairs of PLP-trees, i.e., those that take a distance or similarity matrix as input.

These methods included spectral clustering, affinity propagation, and agglomerative nesting (or AGNES, for short). Spectral clustering (Ng, Jordan, and Weiss 2002) takes the similarity matrix as input, computes the spectrum of it to reduce dimensionality, and clusters the result in fewer dimensions with another algorithm such as K-means. Affinity propagation (Frey and Dueck 2007) passes messages between data points to find representative data points, called *exemplars*, to perform clustering. Lastly, AGNES (Rokach and Maimon 2005) is a bottom-up hierarchical clustering method that starts with viewing each data point per se as a cluster and repeatedly finds two closest clusters and merges them. Based on how the distance is measured between two clusters of data points, AGNES provides three different variants. Given two clusters C_i and C_j of PLP-trees, we define min distance to be $d_{min}(C_i, C_j) = \min_{T_i \in C_i, T_j \in C_j} \tau'(T_i, T_j)$, max distance $d_{max}(C_i, C_j) = \max_{T_i \in C_i, T_j \in C_j} \tau'(T_i, T_j)$, and average distance $d_{avg}(C_i, C_j) = \frac{1}{|C_i| \cdot |C_j|} \sum_{T_i \in C_i} \sum_{T_j \in C_j} \tau'(T_i, T_j)$. AGNES is called *single-linkage*, if d_{min} is used as the distance measure; *complete-linkage*, if d_{max} is used; and *average-linkage*, if d_{avg} is used.

We now define the the Dunn index (DI) and the Davies-Bouldin index (DBI) which we'll use to evaluate cluster quality. The Dunn index is a worst case measurement of the ratio of the minimum intercluster distance to the maximum intracluster distance:

$$DI = \min_{1 \leq i \leq k} \left\{ \min_{j \neq i} \left(\frac{d_{avg}(C_i, C_j)}{\max_{1 \leq l \leq k} (avg(C_l))} \right) \right\}$$

where k is the number of clusters and $avg(C_l)$ is the intracluster distance for cluster l, $avg(C_l) = \frac{1}{\binom{|C_l|}{2}} \sum_{T_i, T_j \in \binom{C_l}{2}} \tau'(T_i, T_j)$. The Davies-Bouldin index is similar, but differs in that it finds a worst case ratio for every cluster and takes the average:

$$DBI = \frac{1}{k} \sum_{i=1}^{k} \max_{j \neq i} \left(\frac{avg(C_i) + avg(C_j)}{d_{avg}(C_i, C_j)} \right)$$

Intuitively, both cluster indices measure the extent to which a set of clusters minimize overall intracluster distances while also maximizing intercluster distances.

Experimental Results

To test ***PlpDis***-powered clustering, we applied the selected algorithms to the task of clustering PLP-forests, consisting of random mixtures of UI-UP, UI-CP, and CI PLP-trees, learned from a Car Evaluation dataset[1] using Liu

Table 1: The *DI* to *DBI* ratio for each clustering

Forest Size	100	500	1000	2,500	5,000	10,000
# Clusters	4	6	8	14	17	24
Spectral	0.090	0.386	0.387	0.332	DNF	DNF
Aff. Prop.	0.337	0.376	0.342	0.317	0.217	0.238
AGNES-A	2.628	**3.534**	2.172	1.555	1.196	1.323
AGNES-S	2.876	2.157	2.172	**2.857**	**1.888**	**2.034**
AGNES-C	**3.393**	2.894	**2.245**	1.176	0.884	0.886

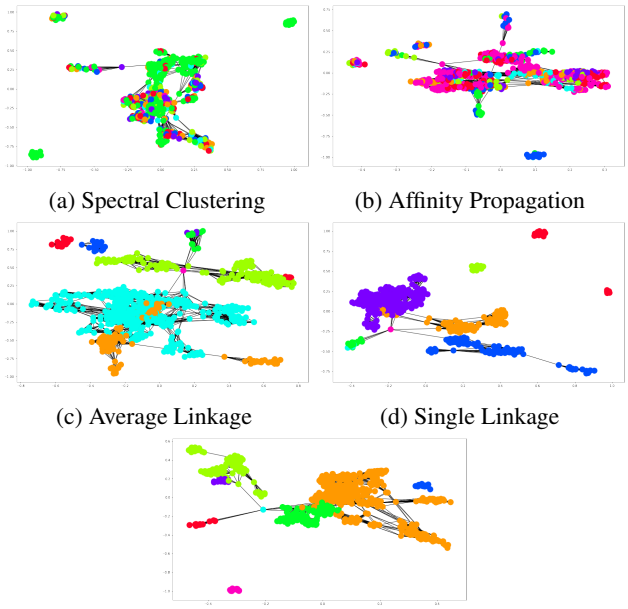
and Truszczynski's greedy algorithm (Liu and Truszczynski 2019). In the experiment we learned forests of sizes 100, 500, 1,000, 2,500, 5,000, and 10,000 with each tree learning from 100 examples. To evaluate cluster quality we used a ratio of DI to DBI, DI/DBI, since higher is better for DI and lower is better for DBI. In addition to this quantitative measure, to visually assess cluster quality we constructed KNN-graphs from the distance matrix and then colored vertices according to each algorithm's cluster assignments. The reasoning for this representation follows from the fact that the distance matrix itself can be seen as a fully-connected graph. By considering only the K nearest neighbors for each tree instead, we see that clusters emerge based on connectedness. After testing multiple values of K, we set $K = 10$ since it provided the best visualizations of the clusters for this experiment. Note that since affinity propagation is the only algorithm that doesn't take the number of clusters k as input, we simply run it first and provide the k it finds to the other algorithms, but k can also be chosen via search.

The results of these tests are included in Table 1, where the best score is bolded in each column. Note that "DNF" in the table for spectral clustering indicates that the algorithm did not finish within a set timeout threshold of 20 minutes. In Figure 2 we show the KNN-graphs from the test with 1000 trees and 8 clusters for each clustering algorithm. Clearly the visual quality of the clusters in these graphs supports the numerical results. Overall, all three variants of AGNES perform better compared to spectral clustering and affinity propagation. AGNES consistently detects the highly-connected vertices (similar trees) in the KNN-graphs, but the other methods do not. This trend continues for all tested forest sizes, with *single-linkage* taking the lead for the largest forest sizes (2,500+).

Future Work

Clustering preference models provides significant insights into decision making and can support many essential applications, such as understanding demographical distributions of user preferences and how those preferences change over time. To this end, we introduced a polynomial algorithm $PlpDis$ that computes Kendall's τ distance between PLP-trees, and presented empirical results using it in combination with existing clustering methods. For future directions, we plan to improve the scalability of our implementation to handle larger numbers of models. We also intend to explore other preferential datasets as well as visualization techniques to improve the interactivity of the clustering results.

[1] www.unf.edu/~xudong.liu/preflearnlib.html



(a) Spectral Clustering (b) Affinity Propagation

(c) Average Linkage (d) Single Linkage

(e) Complete Linkage

Figure 2: 1000 PLP-tree KNN-Graphs (8 clusters)

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